Corrections to the Abelian Born–Infeld Action Arising from Noncommutative Geometry

Lorenzo Cornalba

Institut des Hautes Etudes Scientifiques
cornalba@ihes.fr

January 2000

Abstract

In a recent paper Seiberg and Witten have argued that the full action describing the dynamics of coincident branes in the weak coupling regime is invariant under a specific field redefinition, which replaces the group of ordinary gauge transformations with the one of noncommutative gauge theory. This paper represents a first step towards the classification of invariant actions, in the simpler setting of the abelian single brane theory. In particular we consider a simplified model, in which the group of noncommutative gauge transformations is replaced with the group of symplectic diffeomorphisms of the brane world volume. We carefully define what we mean, in this context, by invariant actions, and rederive the known invariance of the Born–Infeld volume form. With the aid of a simple algebraic tool, which is a generalization of the Poisson bracket on the brane world volume, we are then able to describe invariant actions with an arbitrary number of derivatives.

1 Introduction

The physics of branes with large background magnetic fields is intimately connected, as shown in various works[1] [?, ?], to gauge theories on non–commutative spaces. In particular, it has been shown, in a detailed study [?], that there exists a large freedom in the possible description of the physics of the gauge degrees of freedom which live on the brane world–volume. One is free to choose the non–commutativity parameter on the world–volume, and each possible choice can be reached by a suitable gauge–orbit preserving field redefinition. The most striking feature of the full action describing the brane dynamics at small string coupling is that, regardless of the choice of the non–commutativity parameter, it is, after the above–mentioned field redefinition, invariant in form, in a sense which will be made sharper in the later part of this introductory section. This property of the brane action is highly non–trivial, and does constrain the action in a considerable but not fully understood way. In particular it has been argued in various settings [?, ?] that, if one considers only terms without derivatives, and if one looks at the $U(1)$ single brane theory, then the unique action which is form invariant is the Born–Infeld one, which is known to describe the low energy phenomena of brane physics.

It is of importance to understand how the invariance described in [?] constrains the brane action in the more general non–abelian $U(N)$ context, and also how it constrains higher derivative terms. There has been already some results in this direction [?], but the methods are not systematic, and become of increasing complexity after the first few terms have been constrained. This paper is a first step towards a classification of invariant actions, in a simplified context in which the geometric nature of the problem reduces the task to a manageable one. In particular we will not address the non–abelian case, restricting ourselves to the $U(1)$ theory. Moreover we will work in a simplified setting, relying on a previous note [?, ?] by the author. In particular, we will substitute the group of non–commutative gauge transformations with the simpler group of symplectic diffeomorphisms of the brane world–volume, and we will carefully describe what we mean by invariant actions in this case. With the aid of a simple algebraic tool, which is a generalization of the natural Poisson bracket on the brane world–volume, we will then be able to generate in a simple and powerful way invariant actions with an arbitrary number of derivatives.

Let us note that the classification of invariant actions is intimately tied to a deeper understanding of T–duality in the context of open–string physics [?]. This can be better understood if we toroidally compactify space–time. It is then true that, for some integral values of the magnetic B field, one can consider the brane

[1] For an extensive list of references, we refer the reader to [?].

configuration as a bound state of higher dimensional branes with branes of lower dimension. T–duality then exchanges the two types of branes, and therefore also changes the underlying gauge group. There must therefore exist a (highly non–local) field redefinition which maps gauge orbits of one gauge group to gauge orbits of the other gauge group. Moreover the form of the action must be invariant under T–duality, and therefore the field–redefinition must respect the form of the action. Again, this is a highly non–trivial requirement, and it can be shown [**?**] to be equivalent, using simple Morita equivalence arguments, to the statements described in [**?**].

This paper has the following structure. We conclude this section by recalling the results of [**?**], both to set notation and to clarify what we mean by invariance of the brane action. In section 2 we then briefly recall the work [**?**] and describe the simplified setting within which we shall consider the problem. The invariance of the Born–Infeld action is then shown in section 3, and it is used to give a clear definition, in section **??**, of what we mean by invariant actions within the setting of this paper. In section **??** we finally introduce the generalized bracket and we show how it can be used to construct invariant brane actions with an arbitrary number of derivatives. A few examples with two derivatives are then considered in section **??**. Conclusions and open problems are left for the final section **??**.

Let us then proceed to a quick review of [**?**]. We will work throughout with units such that

$$2\pi\alpha' = 1.$$

Let $M = \mathbb{R}^n$ be the flat space–time manifold, parametrized by coordinates x^a, and with constant background metric and NS two–form given by the matrices g_{ab} and B_{ab} (we will assume that B_{ab} is invertible). The arguments that follow do not rely on supersymmetry considerations, and are valid both in the context of bosonic string theory as well as in the context of Type II superstring theories. We then indicate with n the space–time dimension, with the understanding that $n = 26$ or $n = 10$.

We shall not be interested in the physics of the closed string sector, and we will accordingly leave the geometry of the background space–time manifold fixed. We will, on the other hand, concentrate on the dynamics of open string sector of the theory, by introducing N branes of maximal size – *i.e.* such that the brane world–volume coincides with the space–time manifold M. The dynamical degrees of freedom are then described by a $U(N)$ connection on M. In the weak coupling regime $g_s \to 0$ the interaction of the brane gauge bosons are computed from string theory disk diagrams, and can be reconstructed from a low energy effective action of the general form

$$S = 1g_s \int d^n x \sqrt{\det g_{ab}} \mathrm{Tr}\left(1 + c^{abcd}\omega_{ab}\omega_{cd} + \cdots\right), \tag{1}$$

where

$$\omega = F + B$$

and the coefficients $c^{abcd}, \cdots$ are constructed from the tensor g_{ab} (for example the first coefficient is $14 g^{ac} g^{bd}$) As indicated by the notation, the complete effect of the NS two–form B is obtained by replacing the $U(N)$ field strength F with $\omega = F + B$ in the action.

The above action is defined only up to field redefinitions. The simplest type of redefinition, which had been already considered extensively in the works [**?**, **?**, **?**], are gauge covariant and leave the general form of the action invariant, with the unique effect of changing some of the coefficients. The redefinition is of the form $A_a \to A_a + d^{bc} D_b F_{ac} + \cdots$, where again the coefficients $d^{bc}, \cdots$ are constructed in terms of the metric. A more powerful possible field redefinition has been shown to exist in the recent work [**?**]. The change of variables does not preserve the group of gauge transformations, but on the other hand it substitutes it with the group of gauge transformations of non–commutative gauge theory on the world–volume M. More precisely, there is a transformation $A_a \to \widehat{A}_a$ (which we shall call Seiberg–Witten transformation) preserving gauge orbits such that, in terms of the non–commutative field strength $\widehat{F}$, or, better, of the combination

$$\Omega = \widehat{F} - B,$$

the action reads

$$S = 1G_s \int d^n \sigma \sqrt{\det G_{ab}} \mathrm{Tr}\left(1 + C^{abcd}\Omega_{ab} \star \Omega_{cd} + \cdots\right).$$

In the above, the new metric tensor G_{ab} and string coupling constant G_s are given by

$$\begin{aligned} G &= -B 1g B \\ 1g_s \sqrt{\det B} &= 1G_s \sqrt{\det G}. \end{aligned} \tag{2}$$

Moreover, the coefficients C^{abcd} are obtained starting from the coefficients c^{abcd} and replacing the metric g_{ab} with G_{ab}. Finally, the non–commutativity parameter defining the product $\star$ and the field–strength $\widehat{F}$ is given by[2]

$$\theta = 1B.$$

In the work [?] the transformation $A_a \to \widehat{A}_a$ is determined using the two requirements that it must preserve gauge orbits and that it must be expressible as a power series in θ, with the coefficients of the series being local expressions in the fields. These requirements clearly defines the map up to gauge–covariant local field redefinitions. On the other hand a more precise statement of [?] says that there is, among the possible maps $A_a \to \widehat{A}_a$, one for which the action is form–invariant, in the sense described above.

The problem of invariance has not been analyzed in any detail. We will now describe, in the next sections, a simplified setting which will allow us to tackle the problem in a simple but powerful way.

2 The Simplified Setting

We will, throughout the rest of this paper, work in the simplified context of the abelian $U(1)$ theory. Although this choice does imply a considerable loss of information, we will see that the abelian theory has already a rich structure, and does provide partial information about the non–abelian case.

The second simplification concerns the map $A_a \to \widehat{A}_a$, and follows the author's previous note [?]. In this section we quickly review the results of [?], and we rephrase them in the context of the problem at hand. The starting point of [?] is the observation that the two–form $\omega = B + F$ defines a symplectic structure on M. On one hand ω is clearly closed. Moreover, since we always work perturbatively in F, and since B is invertible, one can take the formal inverse of ω, thus showing that ω is non–degenerate. Therefore, by Darboux's theorem, one can find coordinates σ^i on

[2] We recall that $f \star g = \exp(i2\theta^{ij}\partial_i^f \partial_j^g) f \cdot g$ and that $\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a - i\widehat{A}_a \star \widehat{A}_b + i\widehat{A}_b \star \widehat{A}_a$.

M such that[3]

$$\omega = 12 B_{ij}\, d\sigma^i \wedge d\sigma^j .$$

In these new coordinates, the fluctuations of the field strength F have been replaced by the parallel displacements of the brane, which are described by the coordinate functions $x^a(\sigma)$. Moreover the coordinates σ^i are clearly defined up to symplectic diffeomorphisms of M. The original group of abelian gauge transformations is replaced by the group of symplectomorphisms of (M, ω), and the correspondence between $A_a(x)$ and $x^a(\sigma)$ respects the gauge orbits of the two group actions. One is therefore in a situation similar to the one considered in [?], with the simplifying difference that the group of non–commutative gauge transformations is replaced by the group of symplectomorphisms of the brane world–volume.

To make contact with the notation of the previous section one defines the Poisson bracket $\{,\}$ with respect to the symplectic structure on M

$$\{f, g\} = (1\omega)^{\alpha\beta}\, \partial_\alpha f\, \partial_\beta g.$$

The above formula is particularly simple if one uses the symplectic σ–coordinates, and it then reads

$$\{f, g\} = \theta^{ij} \partial_i f\, \partial_j g.$$

If one defines the non–commutative gauge potential $\widehat{A}_a$ by

$$x^a(\sigma) = \sigma^a + \theta^{ab} \widehat{A}_b(\sigma)$$

and the corresponding field strength by[4]

$$\widehat{F}_{ab} = \partial_a \widehat{A}_b - \partial_b \widehat{A}_a + \left\{ \widehat{A}_a, \widehat{A}_b \right\},$$

[3]We will use the following general conventions concerning coordinate systems and indices. A general coordinate system on M will be denoted by ξ^α, and in general will have Greek indices $\alpha, \beta, \cdots$. The fixed coordinate system x^a will be called flat, and will have in general roman indices $a, b, \cdots$. Finally coordinate systems σ^i for which the two–form ω has constant coefficients B_{ij} will be called symplectic, and will have roman indices $i, j, \cdots$.

[4]This is clearly an exception to the index convention, which is forced by the notation.

one finds that

$$\{x_a, x_b\} = \Omega_{ab} = \widehat{F}_{ab} - B_{ab},$$

where we have lowered the index on the coordinate function x^a using B_{ab}

$$x_a = B_{ab}x^b.$$

We have quickly reviewed the results of [?] and we are therefore in a position, given the above notation, to rephrase the meaning of the invariance of the action given in section 1 in this new framework, starting from the simple invariance of the Born–Infeld volume form.

3 Invariance of the Born–Infeld Volume Form

In this section we prove the exact invariance of the Born–Infeld volume form under the change of coordinates described in the previous section. Before we do so, let us though clarify one point of notation.

In order to limit the number of symbols, we will use, as a general rule, the same letter to indicate an abstract tensor, and its components in a specific coordinate system, and we will rely on our index convention to distinguish among coordinate systems. In some cases though this might be confusing, given the standard notation in the subject. For example the metric tensor $g = g_{ab}dx^a \otimes dx^b$ reads, in a symplectic coordinate system (recall $G_{ab} = B_{ac}B_{bd}g^{cd}$)

$$g = g_{ab}\partial_i x^a \partial_j x^b \, d\sigma^i \otimes d\sigma^j = G^{ab}\partial_i x_a \partial_j x_b \, d\sigma^i \otimes d\sigma^j.$$

Following the general rule, we could use the symbol g_{ij} for $G^{ab}\partial_i x_a \partial_j x_b$. We will not do this, and we will reserve the letter g_{ab} for the constant metric in the flat coordinate system, and will always write $G^{ab}\partial_i x_a \partial_j x_b$ when using σ–coordinates. Similarly we will use B_{ij} instead of ω_{ij}.

With this in mind, let us consider the Born–Infeld volume form

$$\Phi = 1 g_s d^n\xi \sqrt{\det(g + B + F)} = 1 g_s d^n\xi \sqrt{\det(g + \omega)}.$$